

BRANDT EXTENSIONS AND PRIMITIVE TOPOLOGICAL INVERSE SEMIGROUPS

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ABSTRACT. In the paper we study (countably) compact and (absolutely) H -closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

In this paper all spaces are Hausdorff.

A semigroup is a non-empty set with a binary associative operation. A semigroup S is called *inverse* if for any $x \in S$ there exists a unique $y \in S$ such that $x \cdot y \cdot x = x$ and $y \cdot x \cdot y = y$. Such an element y in S is called *inverse* to x and denoted by x^{-1} . The map defined on an inverse semigroup S which maps to any element x of S its inverse x^{-1} is called the *inversion*.

A *topological semigroup* is a Hausdorff topological space with a jointly continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. A *topological group* is a topological space with a continuous group operation and an inversion. We observe that the inversion on a topological inverse semigroup is a homeomorphism (see [6, Proposition II.1]). A Hausdorff topology τ on a (inverse) semigroup S is called (*inverse*) *semigroup* if (S, τ) is a topological (inverse) semigroup.

Further we shall follow the terminology of [2, 3, 7, 17, 20]. If S is a semigroup, then by $E(S)$ we denote the band (the subset of idempotents) of S , and by S^1 [S^0] we denote the semigroup S with the adjoined unit [zero] (see [17, p. 2]). Also if a semigroup S has zero 0_S , then for any $A \subseteq S$ we denote $A^* = A \setminus \{0_S\}$. If Y is a subspace of a topological space X and $A \subseteq Y$, then by $\text{cl}_Y(A)$ we denote the topological closure of A in Y . The set of positive integers is denoted by \mathbb{N} .

If E is a semilattice, then the semilattice operation on E determines the partial order \leq on E :

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

This order is called *natural*. An element e of a partially ordered set X is called *minimal* if $f \leq e$ implies $f = e$ for $f \in X$. An idempotent e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal element in $E(S)$ (in $(E(S))^*$).

Let S be a semigroup with zero and I_λ be a set of cardinality $\lambda \geq 1$. On the set $B_\lambda(S) = (I_\lambda \times S \times I_\lambda) \cup \{0\}$ we define the semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in I_\lambda$ and $a, b \in S$. If $S = S^1$ then the semigroup $B_\lambda(S)$ is called the *Brandt λ -extension of the semigroup S* [9]. Obviously,

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$\mathcal{J} = \{0\} \cup \{(\alpha, \mathcal{O}, \beta) \mid \mathcal{O} \text{ is the zero of } S\}$ is an ideal of $B_\lambda(S)$. We put $B_\lambda^0(S) = B_\lambda(S)/\mathcal{J}$ and we shall call $B_\lambda^0(S)$ the *Brandt λ^0 -extension of the semigroup S with zero* [12]. Further, if $A \subseteq S$ then we shall denote $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A\}$ if A does not contain zero, and $A_{\alpha,\beta} = \{(\alpha, s, \beta) \mid s \in A \setminus \{0\}\} \cup \{0\}$ if $0 \in A$, for $\alpha, \beta \in I_\lambda$. If \mathcal{I} is a trivial semigroup (i.e. \mathcal{I} contains only one element), then by \mathcal{I}^0 we denote the semigroup \mathcal{I} with the adjoined zero. Obviously, for any $\lambda \geq 2$ the Brandt λ^0 -extension of the semigroup \mathcal{I}^0 is isomorphic to the semigroup of $I_\lambda \times I_\lambda$ -matrix units and any Brandt λ^0 -extension of a semigroup with zero contains the semigroup of $I_\lambda \times I_\lambda$ -matrix units. Further by B_λ we shall denote the semigroup of $I_\lambda \times I_\lambda$ -matrix units and by $B_\lambda^0(1)$ the subsemigroup of $I_\lambda \times I_\lambda$ -matrix units of the Brandt λ^0 -extension of a monoid S with zero. A completely 0-simple inverse semigroup is called a *Brandt semigroup* [20]. A semigroup S is a Brandt semigroup if and only if S is isomorphic to a Brandt λ -extension $B_\lambda(G)$ of some group G [20, Theorem II.3.5].

A non-trivial inverse semigroup is called a *primitive inverse semigroup* if all its non-zero idempotents are primitive [20]. A semigroup S is a primitive inverse semigroup if and only if S is an orthogonal sum of Brandt semigroups [20, Theorem II.4.3].

Green's relations \mathcal{L} , \mathcal{R} and \mathcal{H} on a semigroup S are defined by:

$$\begin{aligned} a\mathcal{L}b & \text{ if and only if } a \cup Sa = b \cup Sb; \\ a\mathcal{R}b & \text{ if and only if } a \cup aS = b \cup bS; \text{ and} \\ \mathcal{H} & = \mathcal{L} \cap \mathcal{R}, \end{aligned}$$

for $a, b \in S$. For details about Green's relations see [3, § 2.1] or [8]. We observe that two non-zero elements (α_1, s, β_1) and (α_2, t, β_2) of a Brandt semigroup $B_\lambda(G)$, $s, t \in G$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in I_\lambda$, are \mathcal{H} -equivalent if and only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ (see [20, p. 93]).

By \mathcal{S} we denote some class of topological semigroups.

Definition 1 ([9, 22]). A semigroup $S \in \mathcal{S}$ is called *H-closed in \mathcal{S}* , if S is a closed subsemigroup of any topological semigroup $T \in \mathcal{S}$ which contains S as a subsemigroup. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *H-closed*.

Definition 2 ([10, 23]). A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed in the class \mathcal{S}* if any continuous homomorphic image of S into $T \in \mathcal{S}$ is H-closed in \mathcal{S} . If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *absolutely H-closed*.

A semigroup S is called *algebraically closed in \mathcal{S}* if S with any semigroup topology τ is H-closed in \mathcal{S} and $(S, \tau) \in \mathcal{S}$ [9]. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *algebraically closed*. A semigroup S is called *algebraically h-closed in \mathcal{S}* if S with the discrete topology \mathfrak{d} is absolutely H-closed in \mathcal{S} and $(S, \mathfrak{d}) \in \mathcal{S}$. If \mathcal{S} coincides with the class of all topological semigroups, then the semigroup S is called *algebraically h-closed*.

Absolutely H-closed semigroups and algebraically h-closed semigroups were introduced by Stepp in [23]. There they were called *absolutely maximal* and *algebraic maximal*, respectively.

Definition 3 ([9]). Let λ be a cardinal ≥ 1 and $(S, \tau) \in \mathcal{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that

- a) $(B_\lambda(S), \tau_B) \in \mathcal{S}$; and
- b) $\tau_B|_{(\alpha, S^1, \alpha)} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ -extension of (S, τ) in \mathcal{S}* . If \mathcal{S} coincides with the class of all topological semigroups, then $(B_\lambda(S), \tau_B)$ is called a *topological Brandt λ -extension of (S, τ)* .

Definition 4 ([12]). Let \mathcal{S}_0 be some class of topological semigroups with zero. Let λ be a cardinal ≥ 1 and $(S, \tau) \in \mathcal{S}_0$. Let τ_B be a topology on $B_\lambda^0(S)$ such that

- a) $(B_\lambda^0(S), \tau_B) \in \mathcal{S}_0$;
- b) $\tau_B|_{(\alpha, S, \alpha) \cup \{0\}} = \tau$ for some $\alpha \in I_\lambda$.

Then $(B_\lambda^0(S), \tau_B)$ is called a *topological Brandt λ^0 -extension of (S, τ) in \mathcal{S}_0* . If \mathcal{S}_0 coincides with the class of all topological semigroups, then $(B_\lambda^0(S), \tau_B)$ is called a *topological Brandt λ^0 -extension of (S, τ)* .

Gutik and Pavlyk in [9] proved that the following conditions for a topological semigroup S are equivalent:

- (i) S is an H -closed semigroup in the class of topological inverse semigroups;
- (ii) there exists a cardinal $\lambda \geq 1$ such that any topological Brandt λ -extension of S is H -closed in the class of topological inverse semigroups;
- (iii) for any cardinal $\lambda \geq 1$ every topological Brandt λ -extension of S is H -closed in the class of topological inverse semigroups.

In [10] they showed that the similar statement holds for absolutely H -closed topological semigroups in the class of topological inverse semigroups.

In [12], Gutik and Pavlyk proved the following:

Theorem 5. *Let S be a topological inverse monoid with zero. Then the following conditions are equivalent:*

- (i) S is an (absolutely) H -closed semigroup in the class of topological inverse semigroups;
- (ii) there exists a cardinal $\lambda \geq 1$ such that any topological Brandt λ^0 -extension $B_\lambda^0(S)$ of the semigroup S is (absolutely) H -closed in the class of topological inverse semigroups;
- (iii) for each cardinal $\lambda \geq 1$, every topological Brandt λ^0 -extension $B_\lambda^0(S)$ of the semigroup S is (absolutely) H -closed in the class of topological inverse semigroups.

Also, an example of an absolutely H -closed topological semilattice \mathcal{N} with zero and a topological Brandt λ^0 -extension $B_\lambda^0(\mathcal{N})$ of \mathcal{N} with the following properties:

- (i) $B_\lambda^0(\mathcal{N})$ is an absolutely H -closed semigroup for any infinite cardinal λ ;
- (ii) $B_\lambda^0(\mathcal{N})$ is a σ -compact inverse topological semigroup for any countable cardinal λ ; and
- (iii) $B_\lambda^0(\mathcal{N})$ contains an absolutely H -closed ideal J such that the Rees quotient semigroup $B_\lambda^0(\mathcal{N})/J$ is not a topological semigroup,

were constructed in [12].

We observe that for any topological Brandt λ -extension $B_\lambda(S)$ of a topological semigroup S there exist a topological monoid T with zero and a topological Brandt λ^0 -extension $B_\lambda^0(T)$ of T , such that the semigroups $B_\lambda(S)$ and $B_\lambda^0(T)$ are topologically isomorphic. Algebraic properties of Brandt λ^0 -extensions of monoids with zero and non-trivial homomorphisms between Brandt λ^0 -extensions of monoids with zero, and a category whose objects are ingredients of the construction of Brandt λ^0 -extensions of monoids with zeros were described in [15]. Also, in [13] and [15] was described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0 -extensions of topological monoids with zeros.

In [9] and [18] for every infinite cardinal λ , semigroup topologies on Brandt λ -extensions which preserve an H -closedness and an absolute H -closedness were constructed. An example of a non- H -closed topological inverse semigroup S in the class of topological inverse semigroups such that for any cardinal $\lambda \geq 1$ there exists an absolute H -closed topological Brandt λ -extension of the semigroup S in the class of topological semigroups was constructed in [18].

In this paper we study (countably) compact and (absolutely) H -closed primitive topological inverse semigroups. We describe the structure of compact and countably compact primitive topological inverse semigroups and show that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

Lemma 6. *Let E be a topological semilattice with zero 0 such that every non-zero idempotent of E is primitive. Then every non-zero element of E is an isolated point in E .*

Proof. Let $x \in E^*$. Since E is a Hausdorff topological semilattice, for every open neighbourhood $U(x) \not\ni 0$ of the point x there exists an open neighbourhood $V(x)$ of x such that $V(x) \cdot V(x) \subseteq U(x)$. If x is not an isolated point of E then $V(x) \cdot V(x) \ni 0$ which contradicts to the choice of $U(x)$. This implies the assertion of the lemma. \square

Lemma 7. *Let S be a primitive inverse topological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\}_{i \in \mathcal{A}}$ of topological Brandt semigroups with zeros, i. e. $S = \sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$. Let $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i)$ be a non-zero element of S . Then*

- (i) *there exists an open neighbourhood U of (α_i, g_i, β_i) such that $U \subseteq S_{\alpha_i, \beta_i}^* \subseteq B_{\lambda_i}(G_i)$;*
- (ii) *every non-zero idempotent of S is an isolated point in $E(S)$.*

Proof. (i) Suppose to the contrary that $U \not\subseteq S_{\alpha_i, \beta_i}^* \subseteq B_{\lambda_i}(G_i)$ for any open neighbourhood U of the point (α_i, g_i, β_i) . Since S is a Hausdorff space there exists an open neighbourhood V of the point (α_i, g_i, β_i) such that $0 \notin V$. The continuity of the semigroup operation in S implies that there exists an open neighbourhood W of the point (α_i, g_i, β_i) such that $(\alpha_i, 1_i, \alpha_i) \cdot W \cdot (\beta_i, 1_i, \beta_i) \subseteq V$. Since $W \not\subseteq S_{\alpha_i, \beta_i}^*$, we have that $0 \in V$, a contradiction.

Statement (ii) follows from Lemma 6. \square

Lemma 7 implies

Corollary 8. *Every non-zero \mathcal{H} -class of a primitive inverse topological semigroup S is an open subset in S .*

Lemma 9. *If S is a primitive topological inverse semigroup, then every non-zero \mathcal{H} -class of S is a clopen subset in S .*

Proof. Let $H(e, f)$ be a non-zero \mathcal{H} -class in S for $e, f \in (E(S))^*$, i. e.

$$H(e, f) = \{x \in S \mid x \cdot x^{-1} = e \text{ and } x^{-1} \cdot x = f\}.$$

Since S is a topological inverse semigroup, the maps $\varphi: S \rightarrow E(S)$ and $\psi: S \rightarrow E(S)$ defined by the formulae $\varphi(x) = x \cdot x^{-1}$ and $\psi(x) = x^{-1} \cdot x$ are continuous. By Lemma 6, e and f are isolated points in $E(S)$. Then the continuity of the maps φ and ψ implies the statement of the lemma. \square

The following example shows that the statement of Lemma 9 does not hold for primitive inverse locally compact H -closed topological semigroups.

Example 10. Let \mathbb{Z} be the discrete additive group of integers. We extend the semigroup operation from \mathbb{Z} onto $\mathbb{Z}^0 = \mathbb{Z} \cup \{\infty\}$ as follows:

$$x \cdot \infty = \infty \cdot x = \infty \cdot \infty = \infty, \quad \text{for all } x \in \mathbb{Z}.$$

We observe that \mathbb{Z}^0 is the group with adjoined zero ∞ . We determine a semigroup topology τ on \mathbb{Z}^0 as follows:

- (i) every non-zero element of \mathbb{Z}^0 is an isolated point;
- (ii) the family $\mathcal{B}(\infty) = \{U_n = \{\infty\} \cup \{x \in \mathbb{Z} \mid x \geq n\} \mid n \text{ is a positive integer}\}$ is a base of the topology τ at the point ∞ .

A simple verification shows that (\mathbb{Z}^0, τ) is a primitive inverse locally compact topological semigroup.

Proposition 11. *(\mathbb{Z}^0, τ) is an H -closed topological semigroup.*

Proof. Suppose that \mathbb{Z}^0 is embedded into a topological semigroup T . If $\{n_i\}$ is a net in \mathbb{N} for which $\{-n_i\}$ converges in T to $t \in T \setminus \mathbb{Z}^0$, then the equation $-n_i + (n_i + k) = k$ implies that $t \cdot \infty = k$ for every $k \in \mathbb{N}$ — which is impossible. So \mathbb{Z}^0 is closed in T . \square

Proposition 12. *Every completely 0-simple topological inverse semigroup S is topologically isomorphic to a topological Brandt λ -extension $B_\lambda(G)$ of some topological group G and cardinal $\lambda \geq 1$ in the class of topological inverse semigroups. Furthermore:*

- (i) *any non-zero subgroup of S is topologically isomorphic to G and every non-zero \mathcal{H} -class of S is homeomorphic to G and is a clopen subset in S ;*
- (ii) *the family $\mathcal{B}(\alpha, g, \beta) = \{(\alpha, g \cdot U, \beta) \mid U \in \mathcal{B}_G(e)\}$, where $\mathcal{B}_G(e)$ is a base of the topology at the unity e of G , is a base of the topology at the non-zero element $(\alpha, g, \beta) \in B_\lambda(G)$.*

Proof. Let G be a non-zero subgroup of S . Then by Theorem 3.9 of [3] the semigroup S is isomorphic to the Brandt λ -extension of the subgroup G for some cardinal $\lambda \geq 1$. Since S is a topological inverse semigroup we have that G is a topological group.

(i) Let e be the unity of G . We fix arbitrary $\alpha, \beta, \gamma, \delta \in I_\lambda$ and define the maps $\varphi_{\alpha\beta}^{\gamma\delta}: B_\lambda(G) \rightarrow B_\lambda(G)$ and $\varphi_{\gamma\delta}^{\alpha\beta}: B_\lambda(G) \rightarrow B_\lambda(G)$ by the formulae $\varphi_{\alpha\beta}^{\gamma\delta}(s) = (\gamma, e, \alpha) \cdot s \cdot (\beta, e, \delta)$ and $\varphi_{\gamma\delta}^{\alpha\beta}(s) = (\alpha, e, \gamma) \cdot s \cdot (\delta, e, \beta)$, $s \in B_\lambda(G)$. We observe that $\varphi_{\gamma\delta}^{\alpha\beta}(\varphi_{\alpha\beta}^{\gamma\delta}((\alpha, x, \beta))) = (\alpha, x, \beta)$ and $\varphi_{\alpha\beta}^{\gamma\delta}(\varphi_{\gamma\delta}^{\alpha\beta}((\gamma, x, \delta))) = (\gamma, x, \delta)$ for all $\alpha, \beta, \gamma, \delta \in I_\lambda$, $x \in G$, and hence the restrictions $\varphi_{\alpha\beta}^{\gamma\delta}|_{(\alpha, G, \beta)}$ and $\varphi_{\gamma\delta}^{\alpha\beta}|_{(\gamma, G, \delta)}$ are mutually invertible. Since the maps $\varphi_{\alpha\beta}^{\gamma\delta}$ and $\varphi_{\gamma\delta}^{\alpha\beta}$ are continuous on $B_\lambda(G)$, the map $\varphi_{\alpha\beta}^{\gamma\delta}|_{(\alpha, G, \beta)}: (\alpha, G, \beta) \rightarrow (\gamma, G, \delta)$ is a homeomorphism and the map $\varphi_{\alpha\alpha}^{\gamma\gamma}|_{(\alpha, G, \alpha)}: (\alpha, G, \alpha) \rightarrow (\gamma, G, \gamma)$ is a topological isomorphism. We observe that the subset (α, G, β) of $B_\lambda(G)$ is an \mathcal{H} -class of $B_\lambda(G)$ and (α, G, α) is a subgroup of $B_\lambda(G)$ for all $\alpha, \beta \in I_\lambda$. This completes the proof of assertion (i).

(ii) The statement follows from assertion (i) and Theorem 4.3 of [16]. \square

We observe that Example 10 implies that the statements of Proposition 12 are not true for completely 0-simple inverse topological semigroups. Definition 3 implies that S is a topological Brandt λ -extension $B_\lambda(G)$ of the topological group G .

Gutik and Repovš, in [14], studied the structure of 0-simple countably compact topological inverse semigroups. They proved that any 0-simple countably compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ -extension $B_\lambda(H)$ of a countably compact topological group H in the class of topological inverse semigroups for some finite cardinal $\lambda \geq 1$. This implies Pavlyk's Theorem (see [19]) on the structure of 0-simple compact topological inverse semigroups: *every 0-simple compact topological inverse semigroup is topologically isomorphic to a topological Brandt λ -extension $B_\lambda(H)$ of a compact topological group H in the class of topological inverse semigroups for some finite cardinal $\lambda \geq 1$.*

The following theorem describes the structure of primitive countably compact topological inverse semigroups.

Theorem 13. *Every primitive countably compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Moreover the family*

$$\mathcal{B}(0) = \{S \setminus (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \dots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \mid i_1, i_2, \dots, i_n \in \mathcal{A}, n \in \mathbb{N}\}$$

determines a base of the topology at zero 0 of S .

Proof. By Theorem II.4.3 of [20] the semigroup S is an orthogonal sum of Brandt semigroups and hence S is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of groups G_i . We fix any $i_0 \in \mathcal{A}$. Since S is a topological inverse semigroup, Proposition II.2 [6]

implies that $B_{\lambda_{i_0}}(G_{i_0})$ is a topological inverse semigroup. By Proposition 12, $B_{\lambda_{i_0}}(G_{i_0})$ is a closed subsemigroup of S and hence by Theorem 3.10.4 [7], $B_{\lambda_{i_0}}(G_{i_0})$ is a countably compact 0-simple topological inverse semigroup. Then, by Theorem 2 of [14], the semigroup $B_{\lambda_{i_0}}(G_{i_0})$ is a topological Brandt λ_i -extension of countably compact topological group G_{i_0} in the class of topological inverse semigroups for some finite cardinal $\lambda_{i_0} \geq 1$. This completes the proof of the first assertion of the theorem.

Suppose on the contrary that $\mathcal{B}(0)$ is not a base at zero 0 of S . Then, there exists an open neighbourhood $U(0)$ of zero 0 such that $U(0) \cup (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \dots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \neq S$ for finitely many indexes $i_1, i_2, \dots, i_n \in \mathcal{A}$. Therefore there exists an infinitely family \mathcal{F} of non-zero disjoint \mathcal{H} -classes such that $H \not\subseteq U(0)$ for all $H \in \mathcal{F}$. Let \mathcal{F}_0 be an infinite countable subfamily of \mathcal{F} . We put $W = \bigcup \{H \mid H \in \mathcal{F} \setminus \mathcal{F}_0\}$. Lemma 9 implies that the family $\mathcal{C} = \{U(0), W\} \cup \mathcal{F}_0$ is an open countable cover of S . Simple observation shows that the cover \mathcal{C} does not contains a finite subcover. This contradicts to the countable compactness of S . The obtained contradiction implies the last assertion of the theorem. \square

Since any maximal subgroup of a compact topological semigroup T is a compact subset in T (see [2, Vol. 1, Theorem 1.11]) Theorem 13 implies the following:

Corollary 14. *Every primitive compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$ and the family*

$$\mathcal{B}(0) = \{S \setminus (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \dots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \mid i_1, i_2, \dots, i_n \in \mathcal{A}, n \in \mathbb{N}\}$$

determines a base of the topology at zero 0 of S .

Theorem 15. *Every primitive countably compact topological inverse semigroup S is a dense subsemigroup of a primitive compact topological inverse semigroup.*

Proof. By Theorem 13 the topological semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Since any countably compact topological group G_i is pseudocompact, the Comfort-Ross Theorem (see [4, Theorem 4.1]) implies that the Stone-Ćech compactification $\beta(G_i)$ is a compact topological group and the inclusion mapping f_i of G_i into $\beta(G_i)$ is a topological isomorphism for all $i \in \mathcal{A}$. On the orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ -extensions $B_{\lambda_i}(\beta(G_i))$, $i \in \mathcal{A}$, we determine a topology τ as follows:

- (a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i \cdot U, \beta_i) \mid U \in \mathcal{B}_{\beta(G_i)}(e_i)\}$ is a base of the topology at the non-zero element $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(\beta(G_i))$, where $\mathcal{B}_{\beta(G_i)}(e_i)$ is a base of the topology at the unity e_i of the compact topological group $\beta(G_i)$; and
- (b) the family

$$\mathcal{B}(0) = \{S \setminus (B_{\lambda_{i_1}}(\beta(G_{i_1})) \cup B_{\lambda_{i_2}}(\beta(G_{i_2})) \cup \dots \cup B_{\lambda_{i_n}}(\beta(G_{i_n})))^* \mid i_1, i_2, \dots, i_n \in \mathcal{A}, n \in \mathbb{N}\}$$

determines a base of the topology at zero 0 of $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$.

By Theorem II.4.3 of [20], $\sum_{i \in \mathcal{A}} B_{\lambda_i}(\beta(G_i))$ is a primitive inverse semigroup and simple verifications show that $\sum_{i \in \mathcal{A}} B_{\lambda_i}(\beta(G_i))$ with the topology τ is a compact topological inverse semigroup.

We define a map $f: \sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i) \rightarrow \sum_{i \in \mathcal{A}} B_{\lambda_i}(\beta(G_i))$ as follows:

$$f(0) = 0 \quad \text{and} \quad f((\alpha_i, g_i, \beta_i)) = (\alpha_i, f_i(g_i), \beta_i) \in B_{\lambda_i}(\beta(G_i)) \quad \text{for} \quad (\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i).$$

Simple verifications show that f is a continuous homomorphism. Since $f_i: G_i \rightarrow \beta(G_i)$ is a topological isomorphism, we have that $f: \sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i) \rightarrow \sum_{i \in \mathcal{A}} B_{\lambda_i}(\beta(G_i))$ is a topological isomorphism too. \square

Gutik and Repovš in [14] showed that the Stone-Čech compactification $\beta(T)$ of a 0-simple countably compact topological inverse semigroup T is a 0-simple compact topological inverse semigroup. In this context the following question arises naturally:

Question 16. Is the Stone-Čech compactification $\beta(T)$ of a primitive countably compact topological inverse semigroup T a topological semigroup (a primitive topological inverse semigroup)?

Theorem 17. *Let $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ be a topological inverse semigroup such that*

- (i) S_α is an H -closed (resp., absolutely H -closed) semigroup in the class of topological inverse semigroups for any $\alpha \in \mathcal{A}$; and
- (ii) *there exists an H -closed (resp., absolutely H -closed) subsemigroup T of S in the class of topological inverse semigroups such that $S_\alpha \cdot S_\beta \subseteq T$ for all $\alpha \neq \beta$, $\alpha, \beta \in \mathcal{A}$.*

Then S is an H -closed (resp., absolutely H -closed) semigroup in the class of topological inverse semigroups.

Proof. We consider the case of absolute H -closedness only.

Suppose on the contrary that there exist a topological inverse semigroup G and a continuous homomorphism $h: S \rightarrow G$ such that $h(S)$ is not closed subsemigroup in G . Without loss of generality we can assume that $\text{cl}_G(h(S)) = G$. Thus, by Proposition II.2 of [6], G is a topological inverse semigroup.

Then, $G \setminus h(S) \neq \emptyset$. Let $x \in G \setminus h(S)$. Since S and G are topological inverse semigroups we have that $h(S)$ is an inverse subsemigroup in G and hence $x^{-1} \in G \setminus h(S)$. The semigroup T is an absolutely H -closed semigroup in the class of topological inverse semigroups implies that there exists an open neighbourhood $U(x)$ of the point x in T such that $U(x) \cap h(T) = \emptyset$. Since G is a topological inverse semigroup there exist open neighbourhoods $V(x)$ and $V(x^{-1})$ of the points x and x^{-1} in G , respectively, such that $V(x) \cdot V(x^{-1}) \cdot V(x) \subseteq U(x)$. But $x, x^{-1} \in \text{cl}_G(h(S)) \setminus h(S)$ and since $\{S_\alpha \mid \alpha \in \mathcal{A}\}$ is the family of absolutely H -closed semigroups in the class of topological inverse semigroups, each of the neighbourhoods $V(x)$ and $V(x^{-1})$ intersects infinitely many subsemigroups $h(S_\beta)$ in G , $\beta \in \mathcal{A}$. Hence, $(V(x) \cdot V(x^{-1}) \cdot V(x)) \cap h(T) \neq \emptyset$. This contradicts the assumption that $U(x) \cap h(T) = \emptyset$. The obtained contradiction implies that S is an absolutely H -closed semigroup in the class of topological inverse semigroups.

The proof in the case of H -closeness is similar to the previous one. \square

Theorem 17 implies:

Corollary 18. *Let $S = \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ be an inverse semigroup such that*

- (i) S_α is an algebraically closed (resp., algebraically h -closed) semigroup in the class of topological inverse semigroups for any $\alpha \in \mathcal{A}$; and
- (ii) *there exists an algebraically closed (resp., algebraically h -closed) subsemigroup T of S in the class of topological inverse semigroups such that $S_\alpha \cdot S_\beta \subseteq T$ for all $\alpha \neq \beta$, $\alpha, \beta \in \mathcal{A}$.*

Then S is an algebraically closed (resp., algebraically h -closed) semigroup in the class of topological inverse semigroups.

Theorem 17 implies:

Theorem 19. *Let a topological inverse semigroup S be an orthogonal sum of the family $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of H -closed (resp., absolutely H -closed) topological inverse semigroups with zeros in the class of topological inverse semigroups. Then S is an H -closed (resp., absolutely H -closed) topological inverse semigroup in the class of topological inverse semigroups.*

Corollary 18 implies

Corollary 20. *Let an inverse semigroup S be an orthogonal sum of the family $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ of algebraically closed (resp., algebraically h -closed) inverse semigroups with zeros in the class of topological inverse semigroups. Then S is an algebraically closed (resp., algebraically h -closed) inverse semigroup in the class of topological inverse semigroups.*

Recall [1], a topological group G is called *absolutely closed* if G is a closed subgroup of any topological group which contains G as a subgroup. In our terminology such topological groups are called H -closed in the class of topological groups. In [21] Raikov proved that a topological group G is absolutely closed if and only if it is Raikov complete, i. e. G is complete with respect to the two sided uniformity.

A topological group G is called *h -complete* if for every continuous homomorphism $f: G \rightarrow H$ into a topological group H the subgroup $f(G)$ of H is closed [5]. The h -completeness is preserved under taking products and closed central subgroups [5].

Gutik and Pavlyk in [10] showed that a topological group G is H -closed (resp., absolutely H -closed) in the class of topological inverse semigroups if and only if G is absolutely closed (resp., h -complete).

Theorem 21. *For a primitive topological inverse semigroup S the following assertions are equivalent:*

- (i) *every maximal subgroup of S is absolutely closed;*
- (ii) *the semigroup S with every inverse semigroup topology τ is H -closed in the class of topological inverse semigroups.*

Proof. (i) \Rightarrow (ii) Suppose that a primitive topological inverse semigroup S is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of topological groups G_i in the class of topological inverse semigroups and every topological group G_i is absolutely closed. Then, by Theorem 3 of [9] any topological Brandt λ_i -extension $B_{\lambda_i}(G_i)$ of topological group G_i is H -closed in the class of topological inverse semigroups. Theorem 19 implies that S is an H -closed topological inverse semigroup in the class of topological inverse semigroups.

(ii) \Rightarrow (i) Let G be any maximal non-zero subgroup of S . Since S is a primitive topological inverse semigroup we have that S is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ -extensions $B_{\lambda_i}(G_i)$ of topological groups G_i and hence there exists a topological Brandt λ_{i_0} -extension $B_{\lambda_{i_0}}(G_{i_0})$, $i \in \mathcal{A}$, such that $B_{\lambda_{i_0}}(G_{i_0})$ contains the maximal subgroup G and $B_{\lambda_{i_0}}(G_{i_0})$ is a subsemigroup of S .

Suppose on the contrary that the topological group $G = G_{i_0}$ is not absolutely closed. Then there exists a topological group H which contains G as a dense proper subgroup. For every $i \in \mathcal{A}$ we put

$$H_i = \begin{cases} G_i, & \text{if } i \neq i_0; \\ H, & \text{if } i = i_0. \end{cases}$$

On the orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i)$ of Brandt λ -extensions $B_{\lambda_i}(H_i)$, $i \in \mathcal{A}$, we determine a topology τ_0 as follows:

- (a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i \cdot U, \beta_i) \mid U \in \mathcal{B}_{H_i}(e_i)\}$ is a base of the topology at the non-zero element $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(H_i)$, where $\mathcal{B}_{H_i}(e_i)$ is a base of the topology at the unity e_i of the topological group H_i ; and
- (b) the zero 0 is an isolated point in $(\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$.

By Theorem II.4.3 of [20], $\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i)$ with the topology τ_0 is a topological inverse semigroup. Also we observe that the semigroup $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ with the induced from $(\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$

topology is a topological inverse semigroup which is a dense proper inverse subsemigroup of $(\sum_{i \in \mathcal{A}} B_{\lambda_i}(H_i), \tau_0)$. The obtained contradiction completes the statement of the theorem. \square

Theorem 21 implies

Corollary 22. *For a primitive inverse semigroup S the following assertions are equivalent:*

- (i) *every maximal subgroup of S is algebraically closed in the class of topological inverse semigroups;*
- (ii) *the semigroup S is algebraically closed in the class of topological inverse semigroups.*

Theorem 23. *For a primitive topological inverse semigroup S the following assertions are equivalent:*

- (i) *every maximal subgroup of S is h -complete;*
- (ii) *the semigroup S with every inverse semigroup topology τ is absolutely H -closed in the class of topological inverse semigroups.*

Proof. (i) \Rightarrow (ii) Suppose that a primitive topological inverse semigroup S is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i -extensions $B_{\lambda_i}(G_i)$ of topological groups G_i in the class of topological inverse semigroups and every topological group G_i is h -complete. Then by Theorem 14 of [10] any topological Brandt λ_i -extension $B_{\lambda_i}(G_i)$ of topological group G_i is absolutely H -closed in the class of topological inverse semigroups. Theorem 19 implies that S is an absolutely H -closed topological inverse semigroup in the class of topological inverse semigroups.

(ii) \Rightarrow (i) Let G be any maximal non-zero subgroup of S . Since S is a primitive topological inverse semigroup, S is an orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ -extensions $B_{\lambda_i}(G_i)$ of topological groups G_i . Hence there exists a topological Brandt λ_{i_0} -extension $B_{\lambda_{i_0}}(G_{i_0})$, $i \in \mathcal{A}$, such that $B_{\lambda_{i_0}}(G_{i_0})$ contains the maximal subgroup G and $B_{\lambda_{i_0}}(G_{i_0})$ is a subsemigroup of S .

Suppose on the contrary that the topological group $G = G_{i_0}$ is not h -completed. Then there exist a topological group H and continuous homomorphism $h: G \rightarrow H$ such that $h(G)$ is a dense proper subgroup of H . On the Brandt λ -extension $B_{\lambda_{i_0}}(H)$, we determine a topology τ_H as follows:

- (a) the family $\mathcal{B}(\alpha_{i_0}, g_{i_0}, \beta_{i_0}) = \{(\alpha_{i_0}, g_i \cdot U, \beta_{i_0}) \mid U \in \mathcal{B}_H(e)\}$ is a base of the topology at the non-zero element $(\alpha_{i_0}, g_i, \beta_{i_0}) \in B_{\lambda_{i_0}}(H)$, where $\mathcal{B}_H(e)$ is a base of the topology at the unity e of the topological group H ; and
- (b) the zero 0 is an isolated point in $(B_{\lambda_{i_0}}(H), \tau_H)$.

Then $B_{\lambda_{i_0}}(H)$ is an inverse semigroup and simple verifications show that $B_{\lambda_{i_0}}(H)$ with the topology τ_H is a topological inverse semigroup.

On the orthogonal sum $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ -extensions $B_{\lambda_i}(G_i)$, $i \in \mathcal{A}$, we determine a topology τ_* as follows:

- (a) the family $\mathcal{B}(\alpha_i, g_i, \beta_i) = \{(\alpha_i, g_i \cdot U, \beta_i) \mid U \in \mathcal{B}_{G_i}(e_i)\}$ is a base of the topology at the non-zero element $(\alpha_i, g_i, \beta_i) \in B_{\lambda_i}(G_i)$, where $\mathcal{B}_{G_i}(e_i)$ is a base of the topology at the unity e_i of the topological group G_i ; and
- (b) the zero 0 is an isolated point in $(\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i), \tau_*)$.

By Theorem II.4.3 of [20], $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ is a primitive inverse semigroup and simple verifications show that $\sum_{i \in \mathcal{A}} B_{\lambda_i}(G_i)$ with the topology τ_* is a topological inverse semigroup.

We define the map $f: S \rightarrow B_{\lambda_{i_0}}(H)$ as follows:

$$f(x) = \begin{cases} h(x), & \text{if } x \in B_{\lambda_{i_0}}(G_{i_0}); \\ 0, & \text{if } x \notin B_{\lambda_{i_0}}(G_{i_0}), \end{cases}$$

where 0 is zero of S . Evidently the defined map f is a continuous homomorphism. Then $f(S) = B_{\lambda_{i_0}}(h(G_{i_0}))$ is a dense proper inverse subsemigroup of the topological inverse semigroup $(B_{\lambda_{i_0}}(H), \tau_H)$. The obtained contradiction completes the statement of the theorem. \square

Theorem 23 implies

Corollary 24. *For a primitive inverse semigroup S the following assertions are equivalent:*

- (i) *every maximal subgroup of S is algebraically h -closed in the class of topological inverse semigroups;*
- (ii) *the semigroup S is algebraically h -closed in the class of topological inverse semigroups.*

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